8 Infinite series of real numbers

8.1 The Definition and Basic Examples

The most important application of sequences is the definition of convergence of an infinite series. From the elementary school you have been dealing with addition of numbers. As you know the addition gets harder as you add more and more numbers. For example it would take some time to add

$$S_{100} = 1 + 2 + 3 + 4 + 5 + \cdots + 98 + 99 + 100$$

It gets much easier if you add two of these sums, and pair the numbers in a special way:

$$\begin{aligned} 2\,S_{100} = \quad & 1 + \ \ 2 + \ \ 3 + \ \ 4 + \cdots + 97 + 98 + 99 + 100 \\ & 100 + 99 + 98 + 97 + \cdots + \ \ 4 + \ \ 3 + \ \ 2 + \quad 1 \end{aligned}$$

A straightforward observation that each column on the right adds to 101 and that there are 100 such columns yields that

$$2\,S_{100} = 101 \cdot 100, \qquad \text{that is} \qquad S_{100} = \frac{101 \cdot 100}{2} = 5050.$$

This can be generalized to any natural number n to get the formula

$$S_n = 1 + 2 + 3 + 4 + 5 + \cdots + (n-1) + n = \frac{(n+1)\,n}{2}.$$

This procedure indicates that it would be impossible to find the sum

$$1 + 2 + 3 + 4 + 5 + \cdots + n + \cdots$$

where the last set of $\cdots$ indicates that we continue to add natural numbers.

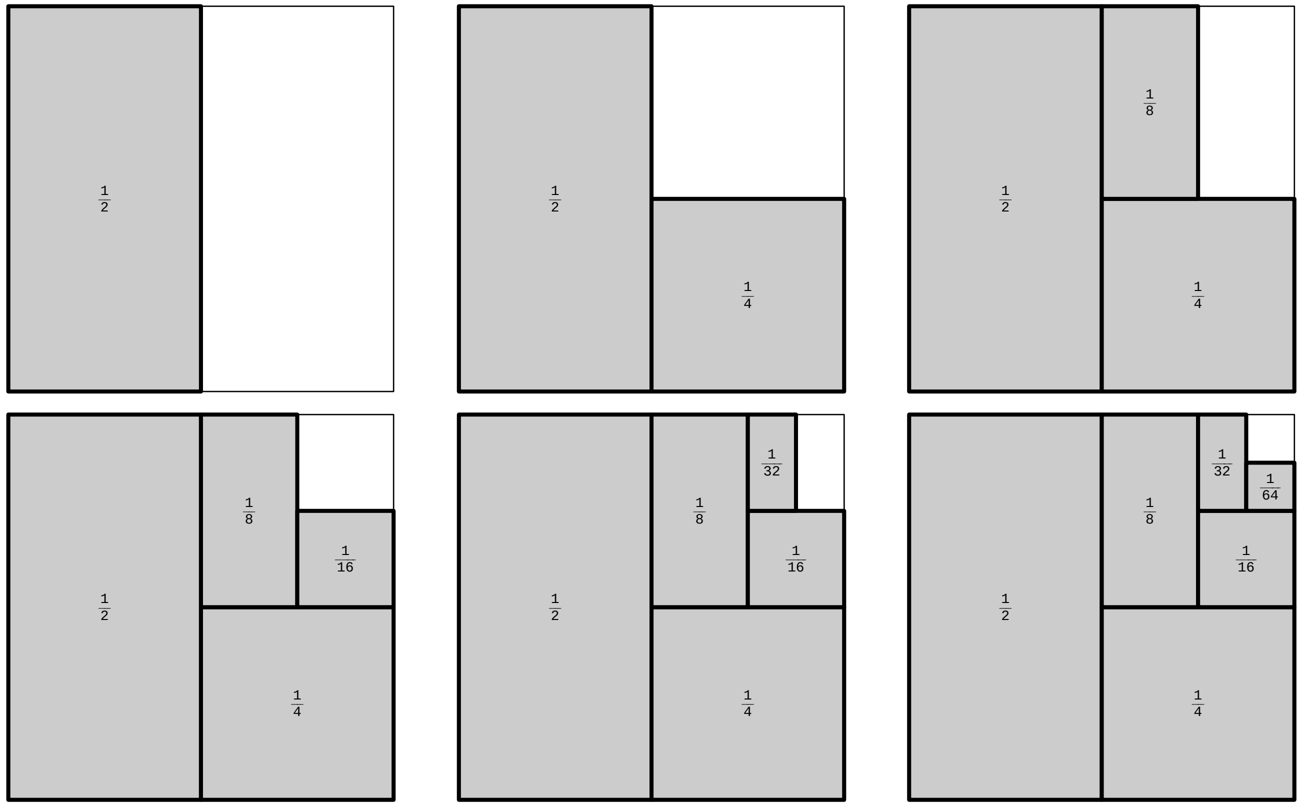
The situation is quite different if we consider the sequence

$$\frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16},\ \ldots,\ \frac{1}{2^n},\ \cdots$$

and start adding more and more consecutive terms of this sequence:

$$\begin{aligned}
&\frac{1}{2} && = 1 - \frac{1}{2} = \frac{1}{2}\\
&\frac{1}{2} + \frac{1}{4} && = 1 - \frac{1}{4} = \frac{3}{4}\\
&\frac{1}{2} + \frac{1}{4} + \frac{1}{8} && = 1 - \frac{1}{8} = \frac{7}{8}\\
&\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} && = 1 - \frac{1}{16} = \frac{15}{16}\\
&\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} && = 1 - \frac{1}{32} = \frac{31}{32}\\
&\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} && = 1 - \frac{1}{64} = \frac{63}{64}
\end{aligned}$$

These sums are nicely illustrated by the following pictures



In this example it seems natural to say that the sum of infinitely many numbers $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ equals 1:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} + \cdots = 1$$

Why does this make sense? This makes sense since we have seen above that as we add more and more terms of the sequence

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^n}, \ldots$$

we are getting closer and closer to 1. Indeed,

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$$

and

$$\lim_{n \to +\infty} \left(1 - \frac{1}{2^n}\right) = 1.$$

This reasoning leads to the definition of convergence of an infinite series:

Definition 8.1.1. Let $\{a_n\}_{n=1}^{+\infty}$ be a given sequence. Then the expression

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is called an infinite series. We often abbreviate it by writing

$$a_1 \ + \ a_2 \ + \ a_3 \ + \ \cdots \ + \ a_n \ + \cdots = \sum_{n=1}^{+\infty} a_n.$$

For each natural number n we calculate the (finite) sum of the first n terms of the series

$$s_n = a_1 \ + \ a_2 \ + \ a_3 \ + \ \cdots \ + \ a_n.$$

We call s_n a partial sum of the infinite series $\displaystyle\sum_{n=1}^{+\infty} a_n$. (Notice that $\{s_n\}_{n=1}^{+\infty}$ is a new sequence.) If the sequence $\{s_n\}_{n=1}^{+\infty}$ converges and if

$$\lim_{n\to+\infty} s_n = S,$$

then the infinite series $\sum_{n=1}^{+\infty} a_n$ is called convergent and we write

$$a_1 \ + \ a_2 \ + \ a_3 \ + \ \cdots \ + \ a_n \ + \cdots = S \qquad \text{or} \qquad \sum_{n=1}^{+\infty} a_n = S.$$

The number S is called the sum of the series.

If the sequence $\{s_n\}_{n=1}^{+\infty}$ does not converge, then the series is called divergent.

In the example above we have

$$a_n = \frac{1}{2^n} = \left(\frac{1}{2}\right)^n,$$
$$s_n = 1 - \frac{1}{2^n} = \frac{2^n - 1}{2^n}$$
$$\lim_{n\to+\infty} \left(1 - \frac{1}{2^n}\right) = 1.$$

Therefore we say that the series

$$\frac{1}{2} \ + \ \frac{1}{4} \ + \ \frac{1}{8} \ + \ \frac{1}{16} \ + \ \cdots \ + \ \frac{1}{2^n} \ + \cdots \ = \sum_{n=1}^{+\infty} \frac{1}{2^n}$$

converges and its sum is 1. We write $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{2^n} = 1$.

In our starting example

$$a_n = n,$$
$$s_n = 1 + 2 + 3 + \cdots + n = \frac{(n+1)n}{2}$$
$$\lim_{n\to+\infty} \frac{(n+1)n}{2} \quad \text{does not exist.}$$

Therefore we say that the series

$$1 \,+\, 2 \,+\, 3 \,+\, 4 \,+\, \cdots \,+\, n \,+ \cdots = \sum_{n=1}^{+\infty} n$$

diverges.

Example 8.1.2 (Geometric Series)**.** Let a and r be real numbers. The most important infinite series is

$$a + a\,r + a\,r^2 + a\,r^3 + \cdots + a\,r^n + \cdots = \sum_{n=0}^{+\infty} a\,r^n \tag{8.1.1}$$

This series is called a geometric series. To determine whether this series converges or not we need to study its partial sums:

$$\begin{aligned} s_0 &= a, & s_1 &= a + a\,r, \\ s_2 &= a + a\,r + a\,r^2, & s_3 &= a + a\,r + a\,r^2 + a\,r^3, \\ s_4 &= a + a\,r + a\,r^2 + a\,r^3 + a\,r^4, & s_5 &= a + a\,r + a\,r^2 + a\,r^3 + a\,r^4 + a\,r^5, \\ &\vdots \\ s_n &= a + a\,r + a\,r^2 + \cdots + a\,r^{n-1} + a\,r^n \\ &\vdots \end{aligned}$$

Notice that we have already studied the special case when $a = 1$ and $r = \dfrac{1}{2}$. In this special case we found a simple formula for s_n and then we evaluated $\lim\limits_{n\to+\infty} s_n$. It turns out that we can find a simple formula for s_n in the general case as well.

First note that the case $a = 0$ is not interesting, since then all the terms of the geometric series are equal to 0 and the series clearly converges and its sum is 0. Assume that $a \neq 0$. If $r = 1$ then $s_n = n\,a$. Since we assume that $a \neq 0$, $\lim\limits_{n\to+\infty} n\,a$ does not exist. Thus for $r = 1$ the series diverges.

Assume that $r \neq 1$. To find a simple formula for s_n, multiply the long formula for s_n above by r to get:

$$\begin{aligned} s_n &= a + a\,r + a\,r^2 + \cdots + a\,r^{n-1} + a\,r^n, \\ r\,s_n &= a\,r + a\,r^2 + \cdots + a\,r^n + a\,r^{n+1}; \end{aligned}$$

now subtract,

$$s_n - r\,s_n = a - a\,r^{n+1},$$

and solve for s_n:

$$s_n = a\,\frac{1 - r^{n+1}}{1 - r}\,.$$

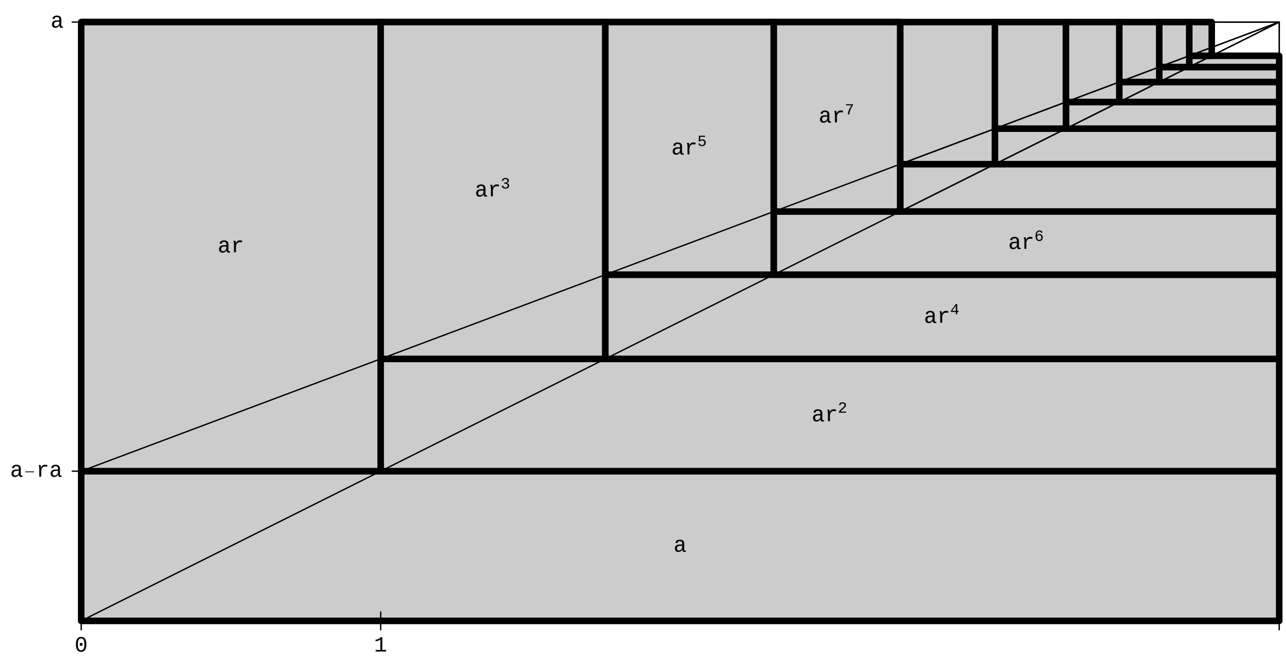
We already proved that if $|r| < 1$, then $\lim_{n\to+\infty} r^{n+1} = 0$. If $|r| \geq 1$, then $\lim_{n\to+\infty} r^{n+1}$ does not exist. Therefore we conclude that

$$\begin{array}{lll} \lim_{n\to+\infty} s_n = \lim_{n\to+\infty} a\,\dfrac{1-r^{n+1}}{1-r} = a\,\dfrac{1}{1-r} & \text{for} & |r|<1, \\ \lim_{n\to+\infty} s_n \quad \text{does not exist} & \text{for} & |r| \geq 1\,. \end{array}$$

In conclusion

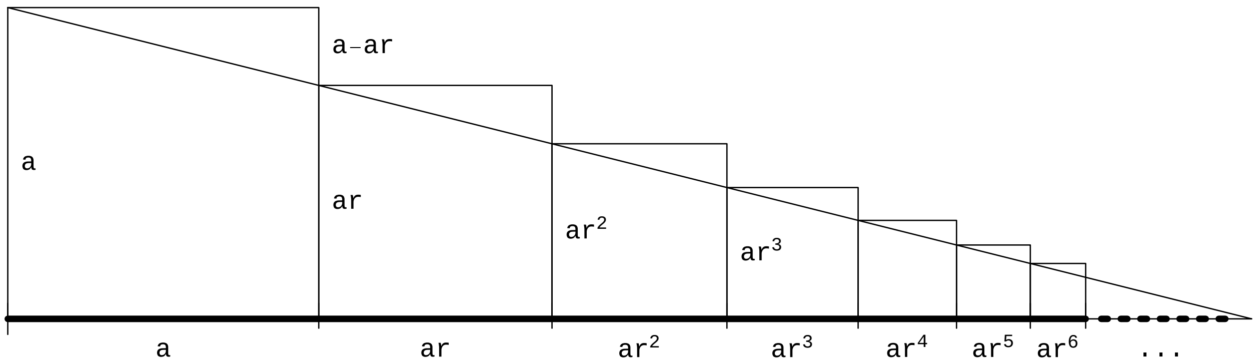
- If $|r| < 1$, then the geometric series $\sum_{n=0}^{+\infty} a\,r^n$ converges and its sum is $a\,\dfrac{1}{1-r}$.
- If $|r| \geq 1$, then the geometric series $\sum_{n=0}^{+\infty} a\,r^n$ diverges.

The following picture illustrates the sum of a geometric series with $a > 0$ and $0 < r < 1$. The width of the rectangle below is $1/(1-r)$ and the height is a. The slopes of the lines shown are $(1-r)a$ and $r(1-r)a$.



In the picture above the terms of a geometric series are represented as areas. As we can see the areas of the terms fill in the rectangle whose area is $a/(1-r)$.

In the picture below we represent the terms of the geometric series by lengths of horizontal line segments. The picture strongly indicates that the total length of infinitely many horizontal line segments is $a/(1-r)$. The reason for this is that by the construction the slope of the hypothenuse of the right triangle in the picture below is $-(1-r)$. Since its vertical leg is a, its horizontal leg must be $a(1-r)$.



Remark 8.1.3. **How to recognize whether an infinite series is a geometric series?**

Consider for example the infinite series $\displaystyle\sum_{n=1}^{+\infty} \frac{\pi^{n+2}}{e^{2n-1}}$. Here $a_n = \dfrac{\pi^{n+2}}{e^{2n-1}}$.

Looking at the formula (8.1.1) we note that the first term of the series is a and that the ratio between any two consecutive terms is r.

For $a_n = \dfrac{\pi^{n+2}}{e^{2n-1}}$ given above we calculate

$$\frac{a_{n+1}}{a_n} = \frac{\dfrac{\pi^{n+1+2}}{e^{2(n+1)-1}}}{\dfrac{\pi^{n+2}}{e^{2n-1}}} = \frac{\pi^{n+3}\, e^{2n-1}}{e^{2n+1}\, \pi^{n+2}} = \frac{\pi}{e^2}\,.$$

Since $\dfrac{a_{n+1}}{a_n}$ is constant, we conclude that the series $\displaystyle\sum_{n=1}^{+\infty} \frac{\pi^{n+2}}{e^{2n-1}}$ is a geometric series with

$$a = a_1 = \frac{\pi^2}{e} \quad \text{and} \quad r = \frac{\pi}{e^2} \quad \text{for all} \quad n = 1, 2, 3, \ldots\,.$$

Since $r = \dfrac{\pi}{e^2} < 1$, we conclude that the sum of this series is

$$\sum_{n=1}^{+\infty} \frac{\pi^{n+2}}{e^{2n-1}} = \frac{\pi^2}{e} \frac{1}{1 - \dfrac{\pi}{e^2}} = \frac{\pi^2}{e} \frac{e^2}{e^2 - \pi} = \frac{\pi^2\, e}{e^2 - \pi}\,.$$

Thus, to verify whether a given infinite series is a geometric series calculate the ratio of the consecutive terms and see whether it is a constant:

$$\sum_{n=1}^{+\infty} a_n \quad \text{for which} \quad \frac{a_{n+1}}{a_n} = r \quad \text{for all} \quad n = 1, 2, 3, \ldots \tag{8.1.2}$$

is a geometric series. In this case $a = a_1$ (the first term of the series).

Example 8.1.4 (Harmonic Series)**.** Harmonic series is the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots = \sum_{n=1}^{+\infty} \frac{1}{n}\,.$$

Again, to explore the convergence of this series we have to study its partial sums:

$$S_1 = 1, \qquad S_2 = 1 + \frac{1}{2},$$
$$S_3 = 1 + \frac{1}{2} + \frac{1}{3}, \qquad S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4},$$
$$S_5 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}, \qquad S_6 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6},$$
$$S_7 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}, \qquad S_8 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8},$$
$$\vdots$$
$$S_n == 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{n}$$
$$\vdots$$

Since $S_{n+1} - S_n = \dfrac{1}{n+1} > 0$ the sequence $\{S_n\}_{n=1}^{+\infty}$ is increasing.

Next we will prove that the sequence $\{S_n\}_{n=1}^{+\infty}$ is not bounded. We will consider only the natural numbers which are powers of 2: $2, 4, 8, \ldots, 2^k, \ldots$. The following inequalities hold:

$$\begin{aligned}
S_2 &= 1 + \frac{1}{2} \geq 1 + \frac{1}{2} && = 1 + 1\,\frac{1}{2}\\
S_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = 1 + \frac{1}{2} + 2\,\frac{1}{4} && = 1 + 2\,\frac{1}{2}\\
S_8 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\\
&\geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = 1 + \frac{1}{2} + 2\,\frac{1}{4} + 4\,\frac{1}{8} && = 1 + 3\,\frac{1}{2}\\
S_{16} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}\\
&\geq 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\\
&= 1 + \frac{1}{2} + 2\,\frac{1}{4} + 4\,\frac{1}{8} + 8\,\frac{1}{16} && = 1 + 4\,\frac{1}{2}
\end{aligned}$$

Continuing this reasoning we conclude that for each $k = 1, 2, 3, \ldots$ the following formula holds:

$$\begin{aligned}
S_{2^k} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{8} + \cdots + \frac{1}{2^{k-1}} + \frac{1}{2^{k-1}+1} + \cdots + \frac{1}{2^k}\\
&\geq 1 + \frac{1}{2} + 2\,\frac{1}{4} + 4\,\frac{1}{8} + 8\,\frac{1}{16} + \cdots + 2^{k-1}\,\frac{1}{2^k} && = 1 + k\,\frac{1}{2}
\end{aligned}$$

Thus

$$S_{2^k} \geq 1 + k\,\frac{1}{2} \quad \text{for all} \quad k = 1, 2, 3, \ldots \tag{8.1.3}$$

This formula implies that the sequence $\{S_n\}_{n=1}^{+\infty}$ is not bounded. Namely, let M be an arbitrary real number. We put $j = \max\bigl\{2\ \text{Floor}(M), 1\bigr\}$. Then

$$j \geq 2\ \text{Floor}(M) > 2(M-1)\,.$$

Therefore,

$$1 + j\,\frac{1}{2} > M\,.$$

Together with the inequality (8.1.3) this implies that

$$S_{2^j} > M\,.$$

Thus for an arbitrary real number M there exists a natural number $n = 2^j$ such that $S_n > M$. This proves that the sequence $\{S_n\}_{n=1}^{+\infty}$ is not bounded and therefore it is not convergent.

In conclusion:

- The harmonic series diverges.

The next example is an example of a series for which we can find a simple formula for the sequence of its partial sums and easily explore the convergence of that sequence. Examples of this kind are called telescoping series.

Example 8.1.5. Prove that the series $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{n(n+1)}$ converges and find its sum.

Solution. We need to examine the series of partial sums of this series:

$$s_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}, \qquad n = 1, 2, 3, \ldots\,.$$

It turns out that it is easy to find the sum s_n if we use the partial fraction decomposition for each of the terms of the series:

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1} \quad \text{for all} \quad k = 1, 2, 3, \ldots\,.$$

Now we calculate:

$$\begin{aligned} s_n &= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n\,(n+1)} \\ &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1}\,. \end{aligned}$$

Thus $s_n = 1 - \dfrac{1}{n+1}$ for all $n = 1, 2, 3, \ldots$. Using the algebra of limits we conclude that

$$\lim_{n\to+\infty} s_n = \lim_{n\to+\infty} \left(1 - \frac{1}{n+1}\right) = 1\,.$$

Therefore the series $\sum_{n=1}^{+\infty} \frac{1}{n\,(n+1)}$ converges and its sum is 1:

$$\sum_{n=1}^{+\infty} \frac{1}{n\,(n+1)} = 1\,.$$

□

Exercise 8.1.6. Determine whether the series is convergent or divergent. If it is convergent, find its sum.

(a) $\sum_{n=1}^{+\infty} 6\left(\frac{2}{3}\right)^{n-1}$ (b) $\sum_{n=1}^{+\infty} \frac{(-2)^{n+3}}{5^{n-1}}$ (c) $\sum_{n=0}^{+\infty} \frac{(\sqrt{2})^n}{2^{n+1}}$ (d) $\sum_{n=1}^{+\infty} \frac{e^{n+3}}{\pi^{n-1}}$

(e) $\sum_{n=1}^{+\infty} \frac{2^{2n-1}}{\pi^n}$ (f) $\sum_{n=1}^{+\infty} \frac{5}{2\,n}$ (g) $\sum_{n=0}^{+\infty} (\sin 1)^n$ (h) $\sum_{n=0}^{+\infty} \frac{2}{n^2+4n+3}$

(i) $\sum_{n=0}^{+\infty} (\cos 1)^n$ (j) $\sum_{n=2}^{+\infty} \frac{2}{n^2-1}$ (k) $\sum_{n=0}^{+\infty} (\tan 1)^n$ (l) $\sum_{n=1}^{+\infty} \ln\left(1+\frac{1}{n}\right)$

A digit is a number from the set $\{0,1,2,3,4,5,6,7,8,9\}$. A decimal number with digits $d_1, d_2, d_3, \ldots, d_n, \ldots$ is in fact an infinite series:

$$0.d_1d_2d_3\ldots d_n\ldots \;=\; \sum_{n=1}^{+\infty} \frac{d_n}{10^n}\,.$$

Therefore each decimal number with digits that repeat leads to a geometric series. We use the following abbreviation:

$$0.\overline{d_1d_2d_3\ldots d_k} = 0.d_1d_2d_3\ldots d_kd_1d_2d_3\ldots d_kd_1d_2d_3\ldots d_kd_1d_2d_3\ldots d_k\ldots$$

Exercise 8.1.7. Express the number as a ratio of integers.

(a) $0.\overline{9} = 0.999\ldots$ (b) $0.\overline{7} = 0.777\ldots$ (c) $0.\overline{712}$ (d) $0.\overline{5432}$

8.2 Basic Properties of Infinite Series

An immediate consequence of the definition of a convergent series is the following theorem

Theorem 8.2.1. *If a series* $\sum_{n=1}^{+\infty} a_n$ *converges, then* $\lim_{n\to+\infty} a_n = 0$.

Proof. Assume that $\sum_{n=1}^{+\infty} a_n$ is a convergent series. By the definition of convergence of a series its sequence of partial sums $\{s_n\}_{n=1}^{+\infty}$ converges to some number S: $\lim_{n\to+\infty} s_n = S$. Then also $\lim_{n\to+\infty} s_{n-1} = S$. Now using the formula

$$a_n = s_n - s_{n-1}, \quad \text{for all} \quad n = 2,3,4,\ldots,$$

and the algebra of limits we conclude that

$$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} s_n - \lim_{n\to+\infty} s_{n-1} = S - S = 0.$$

□

Warning: The preceding theorem cannot be used to conclude that a particular series converges. Notice that in this theorem it is assumed that $\sum_{n=1}^{+\infty} a_n$ is a convergent.

On a positive note: Theorem 8.2.1 can be used to conclude that a given series diverges: If we know that $\lim_{n\to+\infty} a_n = 0$ is not true, then we can conclude that the series $\sum_{n=1}^{+\infty} a_n$ diverges. This is a useful test for divergence.

Theorem 8.2.2 (The Test for Divergence)**.** *If the sequence $\{a_n\}_{n=1}^{+\infty}$ does not converge to 0, then the series $\sum_{n=1}^{+\infty} a_n$ diverges.*

Example 8.2.3. Determine whether the infinite series $\sum_{n=1}^{+\infty} \cos\left(\frac{1}{n}\right)$ converges or diverges.

Solution. Just perform the divergence test:

$$\lim_{n\to+\infty} \cos\left(\frac{1}{n}\right) = 1 \neq 0\,.$$

Therefore the series $\sum_{n=1}^{+\infty} \cos\left(\frac{1}{n}\right)$ diverges. □

Example 8.2.4. Determine whether the infinite series $\sum_{n=1}^{+\infty} \frac{n^{(-1)^n}}{n+1}$ converges or diverges.

Solution. Consider the sequence $\left\{\frac{n^{(-1)^n}}{n+1}\right\}_{n=1}^{+\infty}$:

$$\frac{1}{1\cdot 2}, \frac{2}{3}, \frac{1}{3\cdot 4}, \frac{4}{5}, \frac{1}{5\cdot 6}, \frac{6}{7}, \frac{1}{7\cdot 8}, \frac{8}{9}, \frac{1}{9\cdot 10}, \frac{10}{11}, \frac{1}{11\cdot 12}, \frac{12}{13}, \ldots, \frac{1}{(2k-1)\cdot 2k}, \frac{2k}{2k+1}, \ldots \tag{8.2.1}$$

Without giving a formal proof we can tell that this sequence diverges. In my informal language the sequence (8.2.1) is not constantish since it can not decide whether to be close to 0 or 1.

Therefore the series $\sum_{n=1}^{+\infty} \frac{n^{(-1)^n}}{n+1}$ diverges. □

Remark 8.2.5. The divergence test can not be used to answer whether the series $\sum_{n=1}^{+\infty} \sin\left(\frac{1}{n}\right)$ converges or diverges. It is clear that $\lim_{n\to+\infty} \sin\left(\frac{1}{n}\right) = 0$. Thus we can not use the test for divergence.

Theorem 8.2.6 (The Algebra of Convergent Infinite Series)**.** *Assume that* $\sum_{n=1}^{+\infty} a_n$ *and* $\sum_{n=1}^{+\infty} b_n$ *are convergent series. Let* c *be a real number. Then the series*

$$\sum_{n=1}^{+\infty} c\, a_n, \qquad \sum_{n=1}^{+\infty} (a_n + b_n), \qquad \textit{and} \qquad \sum_{n=1}^{+\infty} (a_n - b_n),$$

are convergent series and the following formulas hold

$$\begin{aligned} \sum_{n=1}^{+\infty} c\, a_n &= c \sum_{n=1}^{+\infty} a_n, \\ \sum_{n=1}^{+\infty} (a_n + b_n) &= \sum_{n=1}^{+\infty} a_n + \sum_{n=1}^{+\infty} b_n, \quad \textit{and} \\ \sum_{n=1}^{+\infty} (a_n - b_n) &= \sum_{n=1}^{+\infty} a_n - \sum_{n=1}^{+\infty} b_n\,. \end{aligned}$$

Remark 8.2.7. The fact that we write $\sum_{n=1}^{+\infty} b_n$ does not necessarily mean that $\sum_{n=1}^{+\infty} b_n$ is a genuine infinite series.

For example, let m be a natural number and assume that $b_n = 0$ for all $n > m$. Then $\sum_{n=1}^{+\infty} b_n = \sum_{n=1}^{m} b_n$. In this case the series $\sum_{n=1}^{+\infty} b_n$ is clearly convergent. If $\sum_{n=1}^{+\infty} a_n$ is a convergent (genuine) infinite series, then Theorem 8.2.6 implies that the infinite series $\sum_{n=1}^{+\infty} (a_n + b_n)$ is convergent and

$$\sum_{n=1}^{+\infty} (a_n + b_n) = \sum_{n=1}^{+\infty} a_n + \sum_{n=1}^{m} b_n\,.$$

This in particular means that the nature of convergence of an infinite series can not be changed by changing finitely many terms of the series.

For example, let m be a natural number. Then:

$$\text{The series} \quad \sum_{n=1}^{+\infty} a_n \quad \text{converges if and only if the series} \quad \sum_{k=1}^{+\infty} a_{m+k} \quad \text{converges.}$$

Moreover, if $\sum\limits_{n=1}^{+\infty} a_n$ converges, then the following formula holds

$$\sum_{n=1}^{+\infty} a_n = \sum_{j=1}^{m} a_j \; + \; \sum_{k=1}^{+\infty} a_{m+k} \, .$$

Example 8.2.8. Prove that the series $\sum\limits_{n=1}^{+\infty}\left(\dfrac{\pi}{n(n+1)} - \dfrac{1}{2^n}\right)$ converges and find its sum.

Exercise 8.2.9. Determine whether the series is convergent or divergent. If a series is convergent find its sum.

(a) $\sum\limits_{n=1}^{+\infty} \dfrac{n}{n+1}$ (b) $\sum\limits_{n=1}^{+\infty} \arctan n$ (c) $\sum\limits_{n=0}^{+\infty} \dfrac{3^n+2^n}{5^{n+1}}$ (d) $\sum\limits_{n=2}^{+\infty}\left(\dfrac{3}{n^2-1} + \dfrac{\pi}{e^n}\right)$

(e) $\sum\limits_{n=0}^{+\infty} \dfrac{e^n+\pi^n}{2^{2n-1}}$ (f) $\sum\limits_{n=1}^{+\infty} n \sin\left(\dfrac{1}{n}\right)$ (g) $\sum\limits_{n=0}^{+\infty} \dfrac{(n+1)^2}{n^2+1}$ (h) $\sum\limits_{n=0}^{+\infty} \left((0.9)^n + (0.1)^n\right)$

Exercise 8.2.10. Express the following sums as ratios of integers and as repeating decimal numbers.

(a) $0.\overline{47} + 0.\overline{5}$ (b) $0.\overline{499} + 0.\overline{47}$ (c) $0.\overline{499} + 0.\overline{503}$

8.3 Comparison Theorems

Warning: All series in the next two sections have positive terms! Do not use the tests from these sections for series with some negative terms.

The convergence of the series in Examples 8.1.2 and 8.1.5 was established by calculating the limits of their partial sums. This is not possible for most series. For example we will soon prove that the series

$$\sum_{n=1}^{+\infty} \frac{1}{n^2}$$

converges. To understand why the sum of this series is exactly $\dfrac{\pi^2}{6}$ you need to take a class about Fourier series, Math 430.

I hope that you have done your homework and that you proved that the series

$$\sum_{n=2}^{+\infty} \frac{1}{n^2-1}$$

converges and that you found its sum. If you didn't here is a way to do it: (It turns out that this is a telescoping series.)

Let

$$S_n = \frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \cdots + \frac{1}{n^2-1} \, .$$

Since $S_{n+1} - S_n = \dfrac{1}{(n+1)^2-1} > 0$ the sequence $\{S_n\}_{n=2}^{+\infty}$ is increasing.

For each $k = 2, 3, 4, \dots$ we have the following partial fractions decomposition

$$\frac{1}{k^2-1} = \frac{1}{(k-1)(k+1)} = \frac{1}{2}\left(\frac{1}{k-1} - \frac{1}{k+1}\right).$$

Next we use this formula to simplify the formula for the n-th partial sum

$$\begin{aligned} S_n &= \sum_{k=2}^{n} \frac{1}{k^2-1} = \sum_{k=2}^{n} \frac{1}{2}\left(\frac{1}{k-1} - \frac{1}{k+1}\right) = \frac{1}{2}\sum_{k=2}^{n}\left(\frac{1}{k-1} - \frac{1}{k+1}\right) \\ &= \frac{1}{2}\left(\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{n-2} - \frac{1}{n}\right) + \left(\frac{1}{n-1} - \frac{1}{n+1}\right)\right) \\ &= \frac{1}{2}\left(\frac{1}{1} + \frac{1}{2} - \frac{1}{n} - \frac{1}{n+1}\right) \\ &= \frac{1}{2}\left(\frac{3}{2} - \frac{2n+1}{n(n+1)}\right) = \frac{3}{4} - \frac{2n+1}{2n(n+1)}. \end{aligned}$$

Using the algebra of limits we calculate

$$\lim_{n\to+\infty} \frac{2n+1}{2n(n+1)} = \lim_{n\to+\infty} \frac{\dfrac{2n+1}{n^2}}{\dfrac{2n(n+1)}{n^2}} = \lim_{n\to+\infty} \frac{\dfrac{2}{n} + \dfrac{1}{n^2}}{2\dfrac{n+1}{n}} = \frac{0+0}{2\cdot 1} = 0.$$

Therefore, using the algebra of limits again, we calculate

$$\lim_{n\to+\infty} S_n = \frac{3}{4} - 0 = \frac{3}{4}.$$

Clearly $S_n < \dfrac{3}{4}$ for all $n = 2, 3, \dots$.

Now consider the series

$$\sum_{n=1}^{+\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} + \cdots$$

Let

$$T_n = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2}.$$

The fact that $T_{n+1} - T_n = \dfrac{1}{(n+1)^2} > 0$ implies that the sequence $\{T_n\}_{n=1}^{+\infty}$ is increasing.

Since

$$\frac{1}{4} < \frac{1}{3}, \quad \frac{1}{9} < \frac{1}{8}, \quad \frac{1}{16} < \frac{1}{15}, \quad \dots, \quad \frac{1}{n^2} < \frac{1}{n^2-1},$$

We conclude that

$$T_n = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} < 1 + \frac{1}{3} + \frac{1}{8} + \frac{1}{15} + \cdots + \frac{1}{n^2-1} = 1 + S_n < 1 + \frac{3}{4}.$$

Thus $T_n < \frac{7}{4}$ for all $n = 2, 3, 4, \ldots$. Since the sequence $\{T_n\}_{n=1}^{+\infty}$ is increasing and bounded above it converges by Theorem 7.3.4. Thus the series $\sum_{n=1}^{+\infty} \frac{1}{n^2}$ converges and its sum is $< \frac{7}{4}$.

The principle demonstrated in the above example is the core of the following comparison theorem.

Theorem 8.3.1 (The Comparison Test)**.** *Let* $\sum_{n=1}^{+\infty} a_n$ *and* $\sum_{n=1}^{+\infty} b_n$ *be infinite series with positive terms. Assume that*

$$a_n \le b_n \qquad \textit{for all} \qquad n = 1, 2, 3, \ldots .$$

(a) *If* $\sum_{n=1}^{+\infty} b_n$ *converges, then* $\sum_{n=1}^{+\infty} a_n$ *converges and* $\sum_{n=1}^{+\infty} a_n \le \sum_{n=1}^{+\infty} b_n$.

(b) *If* $\sum_{n=1}^{+\infty} a_n$ *diverges, then* $\sum_{n=1}^{+\infty} b_n$ *diverges.*

Sometimes the following comparison theorem is easier to use.

Theorem 8.3.2 (The Limit Comparison Test)**.** *Let* $\sum_{n=1}^{+\infty} a_n$ *and* $\sum_{n=1}^{+\infty} b_n$ *be infinite series with positive terms. Assume that*

$$\lim_{n\to+\infty} \frac{a_n}{b_n} = L.$$

If $\sum_{n=1}^{+\infty} b_n$ *converges, then* $\sum_{n=1}^{+\infty} a_n$ *converges. Or, equivalently, if* $\sum_{n=1}^{+\infty} a_n$ *diverges, then* $\sum_{n=1}^{+\infty} b_n$ *diverges.*

Example 8.3.3. Determine whether the series $\sum_{n=1}^{+\infty} \frac{n+1}{\sqrt{1+n^6}}$ converges or diverges.

Solution. The dominant term in the numerator is n and the dominant term in the denominator is $\sqrt{n^6} = n^3$. This suggests that this series behaves as the convergent series $\sum_{n=1}^{+\infty} \frac{1}{n^2}$. Since we am trying to prove convergence we will take

$$a_n = \frac{n+1}{\sqrt{1+n^6}} \qquad \text{and} \qquad b_n = \frac{1}{n^2}$$

in the Limit Comparison Test. Now calculate:

$$\lim_{n\to+\infty} \frac{\dfrac{n+1}{\sqrt{1+n^6}}}{\dfrac{1}{n^2}} = \lim_{n\to+\infty} \frac{n^2(n+1)}{\sqrt{1+n^6}} = \lim_{n\to+\infty} \frac{\dfrac{n^2(n+1)}{n^3}}{\dfrac{\sqrt{1+n^6}}{n^3}} = \lim_{n\to+\infty} \frac{1+\dfrac{1}{n}}{\sqrt{\dfrac{1}{n^6}+1}} = 1\,.$$

In the last step we used the algebra of limits and the fact that

$$\lim_{n\to+\infty}\sqrt{\frac{1}{n^6}+1}=1$$

which needs a proof by definition.

Since we proved that $\lim\limits_{n\to+\infty}\dfrac{\dfrac{n+1}{\sqrt{1+n^6}}}{\dfrac{1}{n^2}}=1$ and since we know that $\sum\limits_{n=1}^{+\infty}\dfrac{1}{n^2}$ is convergent, the Limit Comparison Test implies that the series $\sum\limits_{n=1}^{+\infty}\dfrac{n+1}{\sqrt{1+n^6}}$ converges. □

In the next theorem we compare an infinite series with an improper integral of a positive function. Here it is presumed that we know how to determine the convergence or divergence of the improper integral involved.

Theorem 8.3.4 (The Integral Test)**.** *Suppose that $x \mapsto f(x)$ is a continuous positive, decreasing function defined on the interval $(0,+\infty)$. Assume that $a_n = f(n)$ for all $n = 1, 2, \ldots$. Then the following statements are equivalent*

(a) *The integral $\int_1^{+\infty} f(x)\,dx$ converges.*

(b) *The series $\sum\limits_{n=1}^{+\infty} a_n$ converges.*

At this point we assume that you are familiar with improper integrals and that you know how to decide whether an improper integral converges or diverges.

We will use this test in two different forms:

- Prove that the integral $\int_1^{+\infty} f(x)\,dx$ converges. Conclude that the series $\sum\limits_{n=1}^{+\infty} a_n$ converges.
- Prove that the integral $\int_1^{+\infty} f(x)\,dx$ diverges. Conclude that the series $\sum\limits_{n=1}^{+\infty} a_n$ diverges.

Example 8.3.5 (Convergence of p-series)**.** Let p be a real number. The p-series $\sum\limits_{n=1}^{+\infty}\dfrac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \le 1$.

Solution. Let $n > 1$. Then the function $x \mapsto n^x$ is an increasing function. Therefore, if $p < 1$, then $n^p < n$. Consequently,

$$\frac{1}{n^p} > \frac{1}{n}, \qquad \text{for all} \quad n > 1 \quad \text{and} \quad p < 1\,.$$

Since the series $\sum_{n=1}^{+\infty}\frac{1}{n}$ diverges, the Comparison Test implies that the series $\sum_{n=1}^{+\infty}\frac{1}{n^p}$ diverges for all $p \leq 1$.

Now assume that $p > 1$. Consider the function $f(x) = \frac{1}{x^p}$, $x > 0$. This function is a continuous, decreasing, positive function. Let me calculate the improper integral involved in the Integral Test for convergence:

$$\begin{aligned}\int_1^{+\infty}\frac{1}{x^p}\,dx &= \lim_{t\to+\infty}\int_1^t \frac{1}{x^p}\,dx = \lim_{t\to+\infty}\frac{1}{1-p}\,\frac{1}{x^{p-1}}\bigg|_1^t \\ &= \frac{1}{1-p}\lim_{t\to+\infty}\left(\frac{1}{t^{p-1}}-1\right) = \frac{1}{1-p}(-1) = \frac{1}{p-1}\end{aligned}$$

Thus this improper integral converges. Notice that the condition $p > 1$ was essential to conclude that $\lim_{t\to+\infty}\frac{1}{t^{p-1}} = 0$. Since $\frac{1}{n^p} = f(n)$ for all $n = 1, 2, 3, \ldots$, the Integral Test implies that the series $\sum_{n=1}^{+\infty}\frac{1}{n^p}$ converges for $p > 1$. □

Remark 8.3.6. We have not proved this for all $p > 1$ the function $f(x) = \frac{1}{x^p}$, $x > 0$, is continuous. One way to prove that for an arbitrary $a \in \mathbb{R}$ the function $x \mapsto x^a$, $x > 0$ is continuous is to use the identity

$$x^a = e^{a\ln x}, \qquad x > 0.$$

This identity shows that the function $x \mapsto x^a$, $x > 0$ is a composition of the function $\exp(x) = e^x$, $x \in \mathbb{R}$ and the function $x \mapsto a\ln x$, $x > 0$. The later function is continuous by the algebra of continuous functions: It is a product of a constant a and a continuous function ln. We proved that exp is continuous. By Theorem **??** a composition of continuous function is continuous. Consequently $x \mapsto x^a$, $x > 0$ is continuous.

One way to prove this is to use the squeeze

$$1 - p(x-1) \leq \frac{1}{x^p} \leq \frac{1}{1+p(x-1)} \quad \text{for all} \quad x > 1 - \frac{1}{p}.$$

For $p = 2, 3, 4, \ldots$, this squeeze can be proved by induction. For other

Exercise 8.3.7. Determine whether the series is convergent or divergent.

(a) $\sum_{n=1}^{+\infty}\frac{1}{n\sqrt{n}}$ (b) $\sum_{n=1}^{+\infty} ne^{-n^2}$ (c) $\sum_{n=1}^{+\infty}\frac{1}{n\ln n}$ (d) $\sum_{n=1}^{+\infty}\frac{\ln n}{n\sqrt{n}}$

(e) $\sum_{n=1}^{+\infty}\frac{1}{n(\ln n)^b}$ (f) $\sum_{n=1}^{+\infty}\frac{1}{n!}$ (g) $\sum_{n=1}^{+\infty}\sin\left(\frac{1}{n}\right)$ (h) $\sum_{n=2}^{+\infty}\frac{1}{n}\sin\left(\frac{1}{n}\right)$

(i) $\sum_{n=1}^{+\infty}\frac{1}{n}\cos\left(\frac{1}{n}\right)$ (j) $\sum_{n=0}^{+\infty}\frac{\pi+e^n}{e+\pi^n}$ (k) $\sum_{n=1}^{+\infty}\frac{n!}{n^n}$ (l) $\sum_{n=0}^{+\infty}\frac{n^2+1}{\sqrt{n^7+n^3+1}}$

For the series in (e) find all numbers b for which the series converges.

Exercise 8.3.8. A digit is a number from the set $\{0,1,2,3,4,5,6,7,8,9\}$. A decimal number with digits $d_1, d_2, d_3, \ldots, d_n, \ldots$ is in fact an infinite series:

$$0.d_1d_2d_3\ldots d_n\ldots \;=\; \sum_{n=1}^{+\infty} \frac{d_n}{10^n}\,.$$

Use a theorem from this section to prove that the series above always converges.

8.4 The Ratio and Root Tests

Warning: All series in this section have positive terms! Do not use the tests from this section for series with negative terms.

In Remark 8.1.3 we pointed out (see (8.1.2)) that a series

$$\sum_{n=1}^{+\infty} a_n \quad \text{for which} \quad \frac{a_{n+1}}{a_n} = r \quad \text{for all} \quad n = 1,2,3,\ldots$$

is a **geometric series**. Consequently, if $|r| < 1$ this series is convergent, and it is divergent if $|r| \geq 1$.

Testing the series $\displaystyle\sum_{n=0}^{+\infty} \frac{1}{3^n - 2^{n+1}}$ using this criteria leads to the ratio

$$\frac{\dfrac{1}{3^{n+1} - 2^{n+2}}}{\dfrac{1}{3^n - 2^{n+1}}} = \frac{3^n - 2^{n+1}}{3^{n+1} - 2^{n+2}} = \frac{3^n\left(1 - 2\left(\dfrac{2}{3}\right)^n\right)}{3^{n+1}\left(1 - 2\left(\dfrac{2}{3}\right)^n\right)} = \frac{1}{3}\;\frac{1 - 2\left(\dfrac{2}{3}\right)^n}{1 - 2\left(\dfrac{2}{3}\right)^{n+1}}$$

which certainly is not constant, but it is "constantish." We propose that series for which the ratio a_{n+1}/a_n is not constant but constantish, should be called "geometrish." The following theorem tells that convergence and divergence of these series is determined similarly to geometric series.

Theorem 8.4.1 (The Ratio Test)**.** *Assume that* $\displaystyle\sum_{n=1}^{+\infty} a_n$ *is a series with positive terms and that*

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = R\,.$$

Then

(a) *If* $R < 1$*, then the series converges.*

(b) *If* $R > 1$*, then the series diverges.*

Another way to recognize a geometric series is:

$$\text{A series} \quad \sum_{n=1}^{+\infty} a_n \quad \text{for which} \quad \sqrt[n]{\frac{a_{n+1}}{a_1}} = r \quad \text{for all} \quad n = 1,2,3,\ldots$$

is a **geometric series**. Consequently, if $|r| < 1$ this series is convergent, and it is divergent if $|r| \geq 1$.

Testing the series $\displaystyle\sum_{n=0}^{+\infty} \left(\frac{1+n}{1+2n}\right)^n$ using this criteria leads to the root

$$\sqrt[n]{\left(\frac{1+n}{1+2n}\right)^n} = \frac{1+n}{1+2n} = \frac{\frac{1}{n}+1}{\frac{1}{n}+2}$$

which certainly is not constant, but it is "constantish."

Theorem 8.4.2 (The Root Test)**.** *Assume that* $\displaystyle\sum_{n=1}^{+\infty} a_n$ *is a series with positive terms and that*

$$\lim_{n\to+\infty} \sqrt[n]{a_n} = R\,.$$

Then

(a) *If* $R < 1$*, then the series converges.*

(b) *If* $R > 1$*, then the series diverges.*

Remark 8.4.3. Notice that in both the ratio test and the root test if the limit $R = 1$ we can conclude neither divergence nor convergence. In this case the test is inconclusive.

Exercise 8.4.4. Determine whether the series is convergent or divergent.

(a) $\displaystyle\sum_{n=2}^{+\infty} \frac{1}{2^n-3}$	(b) $\displaystyle\sum_{n=1}^{+\infty} \left(\frac{n+2}{2n-1}\right)^n$	(c) $\displaystyle\sum_{n=1}^{+\infty} \frac{4^n}{3^{2n-1}}$	(d) $\displaystyle\sum_{n=1}^{+\infty} \frac{n!}{1\cdot 3\cdot 5\cdots(2n-1)}$
(e) $\displaystyle\sum_{n=1}^{+\infty} \frac{3^n n^2}{n!}$	(f) $\displaystyle\sum_{n=1}^{+\infty} e^{-n} n!$	(g) $\displaystyle\sum_{n=1}^{+\infty} \frac{e^{1/n}}{n^2}$	(h) $\displaystyle\sum_{n=1}^{+\infty} \frac{2\cdot 4\cdot 6\cdots(2n)}{1\cdot 3\cdot 5\cdots(2n-1)}$
(i) $\displaystyle\sum_{n=1}^{+\infty} \frac{(n!)^2}{(2n)!}$	(j) $\displaystyle\sum_{n=1}^{+\infty} \frac{2\,n^{2n}}{(3n^2+1)^n}$	(k) $\displaystyle\sum_{n=1}^{+\infty} \frac{2^{3n}}{3^{2n}}$	(l) $\displaystyle\sum_{n=1}^{+\infty} \frac{1}{(\arctan n)^n}$
(m) $\displaystyle\sum_{n=1}^{+\infty} \frac{n^2}{2^n}$	(n) $\displaystyle\sum_{n=1}^{+\infty} \frac{(n+1)^2}{n2^n}$	(o) $\displaystyle\sum_{n=1}^{+\infty} \frac{a^n}{n!}$	(p) $\displaystyle\sum_{n=1}^{+\infty} \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}$

For some of the problems you might need to use tests from previous sections.

8.5 Alternating Infinite Series

In the previous two sections we considered only series with positive terms. In this section we consider series with both positive and negative terms which alternate: positive, negative, positive, etc. Such series are called **alternating series**. For example

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots + (-1)^{n+1}\frac{1}{n} + \cdots = \sum_{n=1}^{+\infty}(-1)^{n+1}\frac{1}{n} \tag{8.5.1}$$

$$1 - 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} - \frac{1}{3} + \frac{1}{7} - \frac{1}{4} + \frac{1}{8} - \frac{1}{5} + \frac{1}{9} - \frac{1}{6} + \cdots = \sum_{n=1}^{+\infty}\frac{4(-1)^{n+1}}{n\big(3+(-1)^{n+1}\big)} \tag{8.5.2}$$

$$2 - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \frac{6}{5} - \frac{7}{6} + \cdots + (-1)^{n+1}\frac{n+1}{n} + \cdots = \sum_{n=1}^{+\infty}(-1)^{n+1}\frac{n+1}{n} \tag{8.5.3}$$

Theorem 8.5.1 (The Alternating Series Test)**.** *If the alternating series*

$$a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n+1}a_n + \cdots = \sum_{n=1}^{+\infty}(-1)^{n+1}a_n$$

satisfies the following two conditions:

(i) $0 < a_{n+1} \leq a_n \quad \text{for all} \quad n = 1, 2, 3, \ldots,$

(ii) $\lim\limits_{n\to+\infty} a_n = 0,$

then the series is convergent.

Proof. Assume that $\{a_n\}_{n=1}^{+\infty}$ is a non-increasing sequence (that is assume that (i) is true) and $\lim\limits_{n\to+\infty} a_n = 0$.

By the definition of convergence for each $\epsilon > 0$ there exists $N(\epsilon)$ such that

$$n \in \mathbb{N}, \quad n > N(\epsilon) \quad \text{implies} \quad |a_n - 0| < \epsilon\,.$$

Since $a_n > 0$, the last implication can be simplified as

$$n \in \mathbb{N}, \quad n > N(\epsilon) \quad \text{implies} \quad a_n < \epsilon\,. \tag{8.5.4}$$

We need to show that the sequence of partial sums $\{s_n\}_{n=1}^{+\infty}$,

$$s_n = a_1 - a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n+1}a_n, \quad n = 1, 2, 3, 4, \ldots\,,$$

is convergent.

First consider the sequence $\{s_{2n}\}_{n=1}^{+\infty}$ of even partial sums. Then

$$s_{2(n+1)} - s_{2n} = a_{2n+1} - a_{2n+2} \geq 0, \quad \text{since by (i)} \quad a_{2n+2} \leq a_{2n+1}\,.$$

Thus the sequence $\{s_{2n}\}_{n=1}^{+\infty}$ is non-decreasing.

Next we compare an arbitrary even partial sum s_{2k} with an arbitrary odd partial sum s_{2j-1}. Assume $j \leq k$, then

$$s_{2k} - s_{2j-1} = \bigl(-a_{2j} + a_{2j+1}\bigr) + \bigl(-a_{2j+2} + a_{2j+3}\bigr) + \cdots + \bigl(-a_{2k-4} + a_{2k-3}\bigr) + \bigl(-a_{2k-2} + a_{2k-1}\bigr) - a_{2k}.$$

Each of the numbers in the parenthesis is negative. Therefore the last sum is negative. That is $s_{2k} \leq s_{2j-1}$ for $j \leq k$.

Assume now that $j > k$, then

$$s_{2j-1} - s_{2k} = \bigl(a_{2k+1} - a_{2k+2}\bigr) + \bigl(a_{2k+3} - a_{2k+4}\bigr) + \cdots + \bigl(a_{2j-5} - a_{2j-4}\bigr) + \bigl(a_{2j-3} - a_{2j-2}\bigr) + a_{2j-1}.$$

Each of the numbers in the parenthesis is positive. Therefore the last sum is positive. That is $s_{2k} \leq s_{2j-1}$ for $j > k$. Thus we conclude that

$$s_{2k} \leq s_{2j-1} \quad \text{for all} \quad j, k = 1, 2, 3, \ldots . \tag{8.5.5}$$

In particular (8.5.5) means that $\{s_{2n}\}_{n=1}^{+\infty}$ is bounded above and that each s_{2j-1}, $j = 1, 2, 3, \ldots$ is an upper bound. Since the sequence $\{s_{2n}\}_{n=1}^{+\infty}$ is also non-decreasing, the Monotone Convergence Theorem, Theorem 7.3.4, implies that $\{s_{2n}\}_{n=1}^{+\infty}$ converges to its least upper bound, call it S. Consequently

$$s_{2k} \leq S \leq s_{2j-1} \quad \text{for all} \quad j, k = 1, 2, 3, \ldots . \tag{8.5.6}$$

For each two consecutive natural numbers $n, n-1$ one of them is even and one is odd. Therefore the inequalities in (8.5.6) imply that

$$|s_n - S| \leq |s_n - s_{n-1}| = a_n \quad \text{for all} \quad n = 1, 2, 3, \ldots . \tag{8.5.7}$$

Let $\epsilon > 0$ be arbitrary. Let $n \in \mathbb{N}$ be such that $n > N(\epsilon)$. Then by (8.5.4) we conclude that

$$a_n < \epsilon \tag{8.5.8}$$

Combining the inequalities (8.5.7) and (8.5.8) we conclude that

$$|s_n - S| < \epsilon\,.$$

Thus we have proved that for each $\epsilon > 0$ there exists $N(\epsilon)$ such that

$$n \in \mathbb{N}, \quad n > N(\epsilon) \quad \text{implies} \quad |s_n - S| < \epsilon\,.$$

This proves that the sequence $\{s_n\}_{n=1}^{+\infty}$ converges and therefore the alternating series converges. □

Example 8.5.2. The series in (8.5.1) is called alternating harmonic series. It converges.

Solution. We verify two conditions of the Alternating Series Test:

$$a_{n+1} \leq a_n \quad \text{since} \quad \frac{1}{n+1} < \frac{1}{n}, \quad \text{for all} \quad n = 1, 2, 3, \ldots,$$

$$\lim_{n \to +\infty} \frac{1}{n} = 0 \quad \text{is easy to prove by definition.}$$

Thus the Alternating Series Test implies that the alternating harmonic series converges. □

Remark 8.5.3. The Alternating Series Test does not apply to the series in (8.5.2) since the sequence of numbers

$$1,\ 1,\ \frac{1}{3},\ \frac{1}{2},\ \frac{1}{5},\ \frac{1}{3},\ \frac{1}{7},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{5},\ \frac{1}{9},\ \frac{1}{6},\ \ldots,\ \frac{4}{n\bigl(3+(-1)^{n+1}\bigr)},\ \ldots$$

is not non-increasing. Further exploration of the series in (8.5.2) would show that it diverges.

The Alternating Series Test does not apply to the series in (8.5.3) since this series does not satisfy the condition (ii):

$$\lim_{n\to+\infty}\frac{n+1}{n}=1\neq 0\,.$$

Again this series is divergent by the Test for Divergence.

Exercise 8.5.4. Determine whether the given series converges or diverges.

(a) $\displaystyle\sum_{n=1}^{+\infty}\cos\left(n\pi+\frac{1}{n}\right)$ (b) $\displaystyle\sum_{n=0}^{+\infty}\sin\left(n\frac{\pi}{2}\right)$ (c) $\displaystyle\sum_{n=1}^{+\infty}\sin\left(n\pi-\frac{1}{n}\right)$

(d) $\displaystyle\sum_{n=1}^{+\infty}\frac{1}{n}\cos\left(n\pi+\frac{1}{n}\right)$ (e) $\displaystyle\sum_{n=1}^{+\infty}\ln\left(1-\frac{(-1)^n}{n}\right)$ (f) $\displaystyle\sum_{n=1}^{+\infty}\frac{1}{n}\sin\left(n\frac{\pi}{2}\right)$

(g) $\displaystyle\sum_{n=1}^{+\infty}\sin\left(n\frac{\pi}{2}+\frac{1}{n}\right)$ (h) $\displaystyle\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n-(-1)^n}$ (i) $\displaystyle\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{2n-(-1)^n}$

Many of the exercises in the next section use the Alternating Series Test for convergence. Do those exercises as well.

8.6 Absolute and Conditional Convergence

In the previous section we proved that the alternating harmonic series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots+(-1)^{n+1}\frac{1}{n}+\cdots=\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{1}{n}\quad\text{converges.}\tag{8.6.1}$$

Later on we will see that the sum of this series is $\ln 2$.

Talking about infinite series in class we have often used the analogy with an infinite column in a spreadsheet and finding its sum. A series with positive and negative terms one can interpret as balancing a checkbook with (infinitely) many deposits and withdrawals. Looking at the alternating harmonic series (8.6.1) we see a sequence of alternating deposits and withdrawals, infinitely many of them. What we proved in the previous section tells that under two conditions on the deposits and withdrawals, although it has infinitely many transactions, this checkbook can be balanced.

Now comes the first surprising fact! Let's calculate how much has been deposited to this account:

$$1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots+\frac{1}{2n-1}+\cdots=\sum_{n=1}^{+\infty}\frac{1}{2n-1}\,.\tag{8.6.2}$$

Applying the Limit Comparison Test with the harmonic series it is easy to conclude this series diverges. Looking at the withdrawals we see

$$-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\cdots-\frac{1}{2n-1}-\cdots=-\frac{1}{2}\sum_{n=1}^{+\infty}\frac{1}{n}\,. \tag{8.6.3}$$

Again this is a divergent series. This is certainly a suspicious situation: Dealing with an account to which an unbounded amount of money has been deposited and an unbounded amount of money has been withdrawn. A simpler way to look at this is to look at the total amount of money that went through this account (one can call this amount the total "activity" of the account):

$$\sum_{n=1}^{+\infty}\left|(-1)^{n+1}\frac{1}{n}\right| = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots+\frac{1}{n}+\cdots \tag{8.6.4}$$

This is the harmonic series which is divergent.

Since we know that an unbounded amount of money has been deposited to this account we might want to get in the spending mood sooner and do two withdrawals after each deposit, keeping the amounts the same:

$$1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\frac{1}{10}-\frac{1}{12}+\frac{1}{7}-\frac{1}{14}-\frac{1}{16}+\cdots \tag{8.6.5}$$

In any real life checking account this might result in an occasional low balance but if the deposits and withdrawals are identical, no mater how you arrange them they should result in the same final balance. Amazingly this is not always the case with infinite series! (This is the second surprising fact!) The series in (8.6.5) also converges but to a different number then the series in (8.6.1). The following calculation indicates that the sum of the series in (8.6.5) is 1/2 of the sum of the alternating harmonic series in (8.6.1).

$$\begin{aligned}
S_{3k} &= 1-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{6}-\frac{1}{8}+\frac{1}{5}-\frac{1}{10}-\frac{1}{12}+\cdots+\frac{1}{2k-1}-\frac{1}{4k-2}-\frac{1}{4k}\\
&=\frac{1}{2}-\frac{1}{4}+\frac{1}{6}-\frac{1}{8}+\frac{1}{10}-\frac{1}{12}+\cdots+\frac{1}{4k-2}-\frac{1}{4k}\\
&=\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots+\frac{1}{2k-1}-\frac{1}{2k}\right)
\end{aligned}$$

This is a remarkable observation: a change of order of summation can change the sum of an infinite series. This feature is closely related to the fact that the total activity of the account expressed in (8.6.4) is a divergent series. This is a motivation for the following definition.

Definition 8.6.1. A convergent series $\displaystyle\sum_{n=1}^{+\infty} a_n$ is called **conditionally convergent** if the series of the absolute values of its terms $\displaystyle\sum_{n=1}^{+\infty} |a_n|$ is divergent.

Definition 8.6.2. A series $\sum_{n=1}^{+\infty} a_n$ is called **absolutely convergent** if the series of the absolute values of its terms $\sum_{n=1}^{+\infty} |a_n|$ is convergent.

Example 8.6.3. Prove that the series

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \cdots + (-1)^{n+1}\frac{1}{n^2} + \cdots = \sum_{n=1}^{+\infty}(-1)^{n+1}\frac{1}{n^2}$$

is absolutely convergent.

Solution. By the definition of absolute convergence we need to determine the convergence of the series

$$\sum_{n=1}^{+\infty}\left|(-1)^{n+1}\frac{1}{n^2}\right| = \sum_{n=1}^{+\infty}\frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \cdots$$

This is a p-series with $p = 2$. Therefore this series converges. (Notice that at the beginning of Section 8.3 we proved that this series converges by comparing it to a telescoping series.) □

Remark 8.6.4. One can interpreted the series in Example 8.6.3 as a checking account with infinitely many alternating deposits and withdrawals. In this case the total activity of the account is a convergent series. Consequently the total amount deposited

$$1 + \frac{1}{9} + \frac{1}{25} + \cdots + \frac{1}{(2n-1)^2} + \cdots = \sum_{n=1}^{+\infty}\frac{1}{(2n-1)^2} \tag{8.6.6}$$

and the total amount withdrawn

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{36} + \cdots + \frac{1}{(2n)^2} + \cdots = \sum_{n=1}^{+\infty}\frac{1}{(2n)^2} = \frac{1}{4}\sum_{n=1}^{+\infty}\frac{1}{n^2} \tag{8.6.7}$$

are both convergent series. As we can see, the total amount withdrawn is 1/4 of the total activity of the account. We mentioned before that (we can not prove it in this course)

$$\sum_{n=1}^{+\infty}\frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \cdots = \frac{\pi^2}{6}.$$

Therefore

$$\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \frac{1}{36} + \cdots = \frac{3}{4}\frac{\pi^2}{6} - \frac{1}{4}\frac{\pi^2}{6} = \frac{1}{2}\frac{\pi^2}{6} = \frac{\pi^2}{12}$$

Theorem 8.6.5. *If a series* $\sum_{n=1}^{+\infty} a_n$ *is absolutely convergent, then it is convergent.*

Proof. Assume that $\sum_{n=1}^{+\infty} a_n$ is absolutely convergent, that is assume that $\sum_{n=1}^{+\infty} |a_n|$ is convergent. Then the algebra of convergent series the series $\sum_{n=1}^{+\infty} 2\,|a_n|$ is convergent. Since $-|a_n| \leq a_n \leq |a_n|$, we conclude that

$$0 \leq a_n + |a_n| \leq 2\,|a_n| \quad \text{for all} \quad n = 1, 2, 3, \ldots$$

By the Comparison Test it follows that the series $\sum_{n=1}^{+\infty} \bigl(a_n + |a_n|\bigr)$ is convergent. The algebra of convergent series implies that the series

$$\sum_{n=1}^{+\infty} \Bigl(\bigl(a_n + |a_n|\bigr) - |a_n| \Bigr) = \sum_{n=1}^{+\infty} a_n$$

is also convergent. □

The following stronger versions of the Ratio and the Root test can be applied to any series to determine whether a series converges absolutely or it diverges.

Theorem 8.6.6 (The Ratio Test)**.** *Let* $\sum_{n=1}^{+\infty} a_n$ *be a series for which* $\lim_{n \to +\infty} \dfrac{|a_{n+1}|}{|a_n|} = R$. *Then*

(a) *If* $R < 1$, *then the series converges absolutely.*

(b) *If* $R > 1$, *then the series diverges.*

Theorem 8.6.7 (The Root Test)**.** *Let* $\sum_{n=1}^{+\infty} a_n$ *be a series for which* $\lim_{n \to +\infty} \sqrt[n]{|a_n|} = R$. *Then*

(a) *If* $R < 1$, *then the series converges absolutely.*

(b) *If* $R > 1$, *then the series diverges.*

Notice that if the root or the ratio test apply to a series, then series either converges absolutely or diverges. This implies that if a series converges conditionally, then either

$$\lim_{n \to +\infty} \frac{|a_{n+1}|}{|a_n|} = 1 \quad \text{or} \quad \lim_{n \to +\infty} \frac{|a_{n+1}|}{|a_n|} \ \text{does not exist,}$$

and also

$$\lim_{n \to +\infty} \sqrt[n]{|a_n|} = 1 \quad \text{or} \quad \lim_{n \to +\infty} \sqrt[n]{|a_n|} \ \text{does not exist.}$$

In other words, the root and the ratio test cannot lead to a conclusion that a series converges conditionally.

It turns out that our only tool which can be used to conclude conditional convergence is the alternating series test.

Exercise 8.6.8. Determine whether the given series converges conditionally, converges absolutely or diverges.

(a) $\displaystyle\sum_{n=0}^{+\infty} \frac{\cos(n\pi)}{n^2+1}$ (b) $\displaystyle\sum_{n=0}^{+\infty} \frac{\sin(n\pi/2)}{n+1}$ (c) $\displaystyle\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ (d) $\displaystyle\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n\sqrt{n}}$

(e) $\displaystyle\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n^p}$ (f) $\displaystyle\sum_{n=1}^{+\infty} (-1)^{n+1}\frac{e^{1/n}}{n}$ (g) $\displaystyle\sum_{n=1}^{+\infty} (-1)^{n+1}\frac{n^n}{n!}$ (h) $\displaystyle\sum_{n=1}^{+\infty} (-1)^{n+1}\frac{\sqrt{n}}{n+1}$

(i) $\displaystyle\sum_{n=2}^{+\infty} \frac{(-1)^{n+1}}{\ln n}$ (j) $\displaystyle\sum_{n=1}^{+\infty} (-1)^{n+1}\frac{\ln n}{n}$ (k) $\displaystyle\sum_{n=1}^{+\infty} (-1)^{n+1}e^{1/n}$ (l) $\displaystyle\sum_{n=1}^{+\infty} (-1)^{n+1}\ln\frac{n+1}{n}$

In problem (e) determine all the values of p for which the series converges absolutely, converges conditionally and diverges.

Exercise 8.6.9. Determine whether the given series converges conditionally, converges absolutely or diverges.

(a) $\displaystyle\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{(\sin n)^2}{n^2}$ (b) $\displaystyle\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{4}{2n+3+(-1)^n}$

(c) $\displaystyle\sum_{n=1}^{+\infty}(-1)^{n+1}\cos\left(\frac{1}{n}\right)$ (d) $\displaystyle\sum_{n=1}^{+\infty}(-1)^{n+1}\sin\left(\frac{1}{n}\right)$

9 Series of functions

9.1 Power Series

The most important series is the **geometric series**:

$$a + a\,r + a\,r^2 + a\,r^3 + \cdots + a\,r^n + \cdots = \sum_{n=0}^{+\infty} a\,r^n\,.$$

If $-1 < r < 1$ the geometric series converges. Moreover, we proved

$$\sum_{n=0}^{+\infty} a\,r^n = a + a\,r + a\,r^2 + a\,r^3 + \cdots + a\,r^n + \cdots = \frac{a}{1-r} \quad \text{for} \quad -1 < r < 1\,. \tag{9.1.1}$$

Replacing r by x and letting $a = 1$ we can rewrite the formula in (9.1.1) as

$$\sum_{n=0}^{+\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \frac{1}{1-x} \quad \text{for} \quad -1 < x < 1\,. \tag{9.1.2}$$

The formula (9.1.2) can be viewed as a representation of the function

$$f(x) = \frac{1}{1-x}, \quad -1 < x < 1,$$

as an infinite series of powers of x: $1 = x^0, x, x^2, x^3, \ldots$:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \sum_{n=0}^{+\infty} x^n \quad \text{for} \quad -1 < x < 1\,.$$